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INFLUENCE OF SHEAR CORRECTION FACTORS IN THE HIGHER ORDER SHEAR DEFORMATION LAMINATED SHELL THEORY

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Abstract—A third-order shell theory based on Reddy's parabolic shear strain distribution is presented. Upon applying the Donnell shallow shell approximation, the present theory leads to Reddy's formulation in the case that the laminate has a Euclidean middle surface and constant principal radii of curvatures. To accommodate the effect of the continuity condition of interlaminar transverse shear stresses, the shear correction factors are introduced to modify the shear strains in the present higher order theory. The shear correction factors are calculated using an iterative formulation based on the "shear strain energy equivalence". The shell solutions are compared with the elasticity solutions to assess the improvement of using the higher order theory, coupled with the shear correction factors, in predicting the structural responses of moderately thick laminates.

INTRODUCTION

The global shear deformation theories, which treat a laminate as a single-layer shell, provide a compromise between accuracy and computational efficiency in predicting the structural responses of laminated structures. The first-order shear deformation theory (FSDT), based on the works of Reissner (1945) and Mindlin (1951) for isotropic plates, has been extensively employed in the analysis of moderately thick laminates. However, the constant transverse shear strain assumption along the thickness direction in the FSDT violates the prescribed boundary traction conditions and continuity requirements of the interlaminar shear stresses. Therefore, shear correction factors are introduced to accommodate the effect of non-uniform shear strain distribution. A large number of works have been devoted in selecting the "exact" or "improved" values of shear correction factors (Whitney, 1973; Noor and Peters, 1989; Huang, 1993).

Several higher order shear deformation theories (HSDT), based on a nonlinear distribution of displacements in the thickness direction, have been developed (Noor and Burton, 1990). One of these theories, with a growing popularity for laminate analysis, is the third-order theory of Reddy (Reddy, 1984; Reddy and Liu, 1985). By imposing the zero shear strain conditions on the lateral surfaces of the laminates, the nine displacement parameters are reduced to five as in the FSDT. The resulting transverse shear strain distribution is parabolic. With zero transverse shear strain on the surfaces and parabolic shear strain distribution in the thickness direction, it is generally thought that Reddy's HSDT does not require shear correction factors.

The enforced parabolic shear strain distribution represents a close approximation to the true shear strain distribution for a shallow single-layer shell. However, for the cases of multilayered laminates, the continuity condition on the interlaminar shear stresses implies a piecewise continuous shear strain distribution in the thickness direction. Reddy's theory, which approximates in-plane displacements up to the cubic order, produces better in-plane responses than does the FSDT. However, Reddy's HSDT does not consistently yield more accurate deflections than the FSDT with shear correction factors equal to 5/6, especially for laminates with a large number of layers (Khdeir and Reddy, 1991). Therefore, it raises a question whether the introduction of shear correction factors will further improve the performance of the higher order shear deformation theory?

The effect of the shear correction factors on the HSDT is examined in this study. First, a shell formulation based on the orthogonal curvilinear coordinates, modified from Reddy's third-order shallow theory, is presented. The present theory is applicable to deeper shells. For the case of a laminate having a Euclidean surface and constant radii of curvatures, the present theory leads to the same shell formulation as Reddy's HSDT when the Donnell shallow shell approach is applied. The shear correction factors are introduced, similarly as in the FSDT, into the HSDT to modify the transverse shear strain distribution. The shear correction factors are determined from the shear strain energy equivalence formulation, in which the strain energy corresponding to the "quasi-three-dimensional" (quasi-3-D) shear stresses is set equal to the transverse shear strain energy based on the shell theory. The quasi-3-D stresses are the transverse stresses calculated from the three-dimensional (3-D) equilibrium equations; these have more accurate values and distribution profiles than the transverse stresses calculated from the constitutive equations.

For curved panels, the transverse shear stresses, obtained directly by integrating the 3-D equilibrium equations, usually do not satisfy either boundary traction conditions or interlaminar stress continuity requirements. In this study, a set of modified 3-D equilibrium equations reduced from the elasticity equations are employed to calculate the quasi-3-D stresses. The resulting transverse stresses nearly satisfy both traction conditions and interlaminar stress continuity requirements.

The values of shear correction factors depend on the loading conditions, boundary conditions, structural geometries and the details of laminations; the values generally are functions of surface coordinates instead of constants (Huang, 1993). Exact values usually can not be determined directly from the shear strain energy equivalence formulation. An iterative version of shear strain energy equivalence formulation is employed to calculate the values of the shear correction factors for the doubly curved laminates. The improvement of the HSDT coupled with shear correction factors in predicting structural responses of laminated shells (including plates) is examined by comparing the shell solutions to the elasticity solutions.

SHELL EQUATIONS

Consider the middle surface of a doubly curved shell defined by a pair of coordinates (ξ_1 and ξ_2) of an orthogonal curvilinear coordinate system (ξ_1, ξ_2, ξ_3). The surface coordinates are also lines of principal curvatures of the middle surface. The shell thickness direction is coincident with the third coordinate ξ_3 (also referred to as z) as shown in Fig. 1. The ξ_3 coordinate (or z) is always taken to be a physical coordinate. The Lamé parameters of the middle surface are denoted as A_1 and A_2. The radii of curvatures along ξ_1 and ξ_2 curves are R_1 and R_2, respectively. The laminated shell is constructed of an arbitrary number (N) of orthotropic layers with the principal material directions of orthotropy oriented at some arbitrary angle with respect to the ξ_1 and ξ_2 coordinates of the shell. The shell considered has a thickness h.

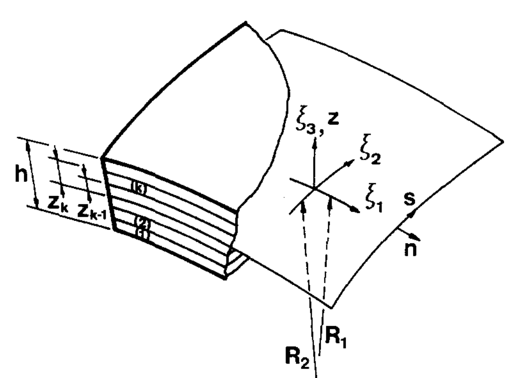

Fig. 1. Doubly curved laminated shell.

The displacements (u, v, w) of an arbitrary point in the shell are expressed as (Librescu *et al.*, 1989)

$$
\begin{aligned}
u(\xi_1,\xi_2,z) &= u^0(\xi_1,\xi_2)-z\theta_1(\xi_1,\xi_2)-z^2\phi_1(\xi_1,\xi_2)-z^3\psi_1(\xi_1,\xi_2)\\
v(\xi_1,\xi_2,z) &= v^0(\xi_1,\xi_2)-z\theta_2(\xi_1,\xi_2)-z^2\phi_2(\xi_1,\xi_2)-z^3\psi_2(\xi_1,\xi_2)\\
w(\xi_1,\xi_2,z) &= w^0(\xi_1,\xi_2).
\end{aligned}
\tag{1}
$$

The relations between transverse shear strains and displacements are

$$
\varepsilon_4 = v_{,3}+w_{,2}/A_2\underline{-v^0/R_2}, \qquad \varepsilon_5 = u_{,3}+w_{,1}/A_1\underline{-u^0/R_1} \tag{2}
$$

in which a comma denotes differentiation with respect to the subscript. In deriving the above expressions, the Love first-order geometric approximation has been invoked (neglecting z/R_1 and z/R_2). It is noted that the Donnell simplification can be accomplished by omitting the terms u^0/R_1 and v^0/R_2 from eqns (2) (underlined terms) (Soedel, 1981). The explicit expressions of ε_4 and ε_5 are

$$
\begin{aligned}
\varepsilon_4 &= -\theta_2+w^0_{,2}/A_2-2z\phi_2-3z^2\psi_2-v^0/R_2\\
\varepsilon_5 &= -\theta_1+w^0_{,1}/A_1-2z\phi_1-3z^2\psi_1-u^0/R_1.
\end{aligned}
\tag{3}
$$

Imposing the zero transverse surface shear strain conditions leads to the following results

$$
\phi_1=\phi_2=0,\quad \psi_1=c\varepsilon_5^0,\quad \psi_2=c\varepsilon_4^0,\quad c=\frac{4}{3h^2} \tag{4}
$$

with

$$
\varepsilon_4^0 = w^0_{,2}/A_2-\theta_2\underline{-v^0/R_2}, \qquad \varepsilon_5^0 = w^0_{,1}/A_1-\theta_1\underline{-u^0/R_1}. \tag{5}
$$

Consequently, the in-plane displacements are of the form

$$
u = u^0-z(\theta_1+cz^2\varepsilon_5^0), \qquad v = v^0-z(\theta_2+cz^2\varepsilon_4^0). \tag{6}
$$

The components of the Lagrangian infinitesimal strain tensor, calculated from the above assumed displacement fields [eqns (1) and (6)], can be expressed as

$$
\begin{aligned}
\varepsilon_i &= \varepsilon_i^0+z(\kappa_i+z^2\lambda_i),\quad i=1,2,6\\
\varepsilon_m &= (1-3cz^2)\varepsilon_m^0,\quad m=4,5
\end{aligned}
\tag{7}
$$

in which ε_m^0 are listed in eqns (5), and ε_i^0, κ_i and λ_i are

$$
\begin{aligned}
\varepsilon_1^0 &= u^0_{,1}/A_1+v^0A_{1,2}/A_1/A_2+w^0/R_1\\
\varepsilon_2^0 &= v^0_{,2}/A_2+u^0A_{2,1}/A_1/A_2+w^0/R_2\\
\varepsilon_6^0 &= v^0_{,1}/A_1+u^0_{,2}/A_2-v^0A_{2,1}/A_1/A_2-u^0A_{1,2}/A_1/A_2\\
\kappa_1 &= -\theta_{1,1}/A_1-\theta_2A_{1,2}/A_1/A_2\\
\kappa_2 &= -\theta_{2,2}/A_2-\theta_1A_{2,1}/A_1/A_2\\
\kappa_6 &= -\theta_{2,1}/A_1-\theta_{1,2}/A_2+\theta_2A_{2,1}/A_1/A_2+\theta_1A_{1,2}/A_1/A_2\\
\lambda_1 &= -c[\varepsilon^0_{5,1}/A_1+\varepsilon_4^0A_{1,2}/A_1/A_2]\\
\lambda_2 &= -c[\varepsilon^0_{4,2}/A_2+\varepsilon_5^0A_{2,1}/A_1/A_2]\\
\lambda_6 &= -c[\varepsilon^0_{4,1}/A_1+\varepsilon^0_{5,2}/A_2-\varepsilon_4^0A_{2,1}/A_1/A_2-\varepsilon_5^0A_{1,2}/A_1/A_2].
\end{aligned}
\tag{8}
$$

To accommodate the effect of the continuity condition of interlaminar shear stresses, following the approach adopted by Kane and Mindlin (1956) and Whitney (1972), the shear strains ε_m are modified as

$$\tilde{\varepsilon}_m = \sqrt{K_m}\varepsilon_m = \sqrt{K_m}(1-3cz^2)\varepsilon_m^0, \quad m = 4,5\,;\text{ not summed} \tag{9}$$

in which K_m are the shear correction factors. A proposed procedure to select the values of the shear correction factors is presented in the next section. The corresponding transverse shear stresses in the kth lamina are

$$(\tilde{\sigma}_m)_k = (\bar{Q}_{mn})_k\tilde{\varepsilon}_n = \sum_n \sqrt{K_n}(\bar{Q}_{mn})_k(1-3cz^2)\varepsilon_n^0, \quad m,n = 4,5. \tag{10}$$

The gross equilibrium equations are obtained using the principle of the virtual work. The resulting shell equations are

$$(A_2N_1)_{,1} - A_{2,1}N_2 + A_{1,2}N_6 + (A_1N_6)_{,2} \underline{+ A_1A_2Q_5/R_1 + (c/R_1)[(A_2P_1)_{,1} - A_{2,1}P_2 + A_{1,2}P_6 + (A_1P_6)_{,2}]} = 0 \tag{11}$$

$$(A_1N_2)_{,2} - A_{1,2}N_1 + A_{2,1}N_6 + (A_2N_6)_{,1} \underline{+ A_1A_2Q_4/R_2 + (c/R_2)[(A_1P_2)_{,2} - A_{1,2}P_1 + A_{2,1}P_6 + (A_2P_6)_{,1}]} = 0 \tag{12}$$

$$\begin{aligned} &A_1A_2N_1/R_1 + A_1A_2N_2/R_2 - (A_2Q_5)_{,1} - (A_1Q_4)_{,2} - c[(A_2P_1)_{,1}/A_1]_{,1} + c(A_{1,2}P_1/A_2)_{,2} \\ &\quad - c[(A_1P_2)_{,2}/A_2]_{,2} + c(A_{2,1}P_2/A_1)_{,1} - c[(A_1P_6)_{,2}/A_1]_{,1} - c[(A_2P_6)_{,1}/A_2]_{,2} \\ &\quad - c(A_{2,1}P_6/A_2)_{,2} - c(A_{1,2}P_6/A_1)_{,1} - A_1A_2(q^+ + q^-) = 0 \end{aligned} \tag{13}$$

$$(A_2M_1)_{,1} - A_{2,1}M_2 + A_{1,2}M_6 + (A_1M_6)_{,2} - A_1A_2Q_5 - c[(A_2P_1)_{,1} - A_{2,1}P_2 + A_{1,2}P_6 + (A_1P_6)_{,2}] = 0 \tag{14}$$

$$(A_1M_2)_{,2} - A_{1,2}M_1 + A_{2,1}M_6 + (A_2M_6)_{,1} - A_1A_2Q_4 - c[(A_1P_2)_{,2} - A_{1,2}P_1 + A_{2,1}P_6 + (A_2P_6)_{,1}] = 0. \tag{15}$$

The equivalent surface tractions q^+ and q^- in eqn (13) are related with prescribed traction p^+ on the surface $z = h/2$ and traction p^- on the other surface $z = -h/2$ as follows:

$$q^+ = p^+\left(1+\frac{h}{2R_1}\right)\left(1+\frac{h}{2R_2}\right), \qquad q^- = p^-\left(1-\frac{h}{2R_1}\right)\left(1-\frac{h}{2R_2}\right). \tag{16}$$

The stress resultants in eqns (11)–(15) are related to ε_i^0, ε_m^0, κ_i and λ_i as

$$\begin{aligned} N_i &= A_{ij}\varepsilon_j^0 + B_{ij}\kappa_j + E_{ij}\lambda_j, \quad i,j = 1,2,6 \\ M_i &= B_{ij}\varepsilon_j^0 + D_{ij}\kappa_j + F_{ij}\lambda_j, \quad i,j = 1,2,6 \\ P_i &= E_{ij}\varepsilon_j^0 + F_{ij}\kappa_j + H_{ij}\lambda_j, \quad i,j = 1,2,6 \\ Q_m &= S_{mn}\varepsilon_n^0, \quad m,n = 4,5. \end{aligned} \tag{17}$$

Stiffnesses A_{ij}, B_{ij}, D_{ij}, E_{ij}, F_{ij}, H_{ij} and S_{mn} are defined as follows:

$$(A_{ij}, B_{ij}, D_{ij}, E_{ij}, F_{ij}, H_{ij}) = \sum_{k=1}^{N}\int_{z_{k-1}}^{z_k}(1,z,z^2,z^3,z^4,z^6)(\bar{Q}_{ij})_k\,\mathrm{d}z, \quad i,j = 1,2,6$$

$$S_{mn} = \sqrt{K_mK_n}A_{mn}$$

$$A_{mn} = \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} (1-3cz^2)^2 (\bar{Q}_{mn})_k \, dz, \quad m, n = 4, 5; \text{not summed,} \tag{18}$$

where $\bar{Q}_{ij}$ are the transformed reduced stiffnesses, and $\bar{Q}_{mn}$ are the transformed shear stiffnesses.

The corresponding boundary conditions along edge ξ_1 = constant are fulfilled by specifying the following displacements or forces

$$\begin{array}{lll}
u^0 & \text{or} & N_1 + \underline{cP_1/R_1} \\
v^0 & \text{or} & N_6 + \underline{cP_6/R_2} \\
w^0 & \text{or} & (A_2M_1)_{,1} - A_{2,1}M_2 + (A_1M_6)_{,2} + A_{1,2}M_6 + cA_1P_{6,2} \\
w^0_{,1} & \text{or} & P_1 \\
\theta_1 & \text{or} & -M_1 + cP_1 \\
\theta_2 & \text{or} & -M_6 + cP_6.
\end{array}$$

The boundary conditions along edge ξ_2 = constant are analogous to the above expressions.

The shallow shell formulation based on the Donnell approximation can be obtained by neglecting the underlined terms from the related expressions. The shallow shell formulations leads to the Reddy formulation for the case in which the laminate has a Euclidean middle surface and constant curvatures. Setting the parameter c equal to zero, the present theory is reduced to the FSDT. It is known that the FSDT yields a solution corresponding to the classical laminate theory when the shear correction factors are assigned a very large value.

CALCULATION OF SHEAR CORRECTION FACTORS

Shear strain energy equivalence formulation

The following set of shear strain energy relations, obtained by equating the strain energy corresponding to the quasi-3-D shear stresses to the shear energy based on the shell theory, are employed to select values for K_4 and K_5

$$\sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} (\sigma_m)_k (\varepsilon_m)_k \, dz = Q_m \varepsilon_m^0, \quad m = 4, 5; \text{not summed} \tag{19}$$

in which $(\sigma_m)_k$ and $(\varepsilon_m)_k$ are the quasi-3-D shear stresses and shear strains in the kth layer. Stresses $(\sigma_m)_k$ in the above equations are obtained by integrating the 3-D equilibrium equations along the thickness direction. For a laminate having arbitrary lay-up, substituting the stress–strain relations into eqns (19) yields the following expressions

$$\begin{aligned}
\sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} (\sigma_4)_k \frac{(\bar{Q}_{55})_k(\sigma_4)_k - (\bar{Q}_{45})_k(\sigma_5)_k}{\Delta_1} &= Q_4 \frac{K_5 A_{55} Q_4 - \sqrt{K_4 K_5} A_{45} Q_5}{K_4 K_5 \Delta_2} \\
\sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} (\sigma_5)_k \frac{(\bar{Q}_{44})_k(\sigma_5)_k - (\bar{Q}_{45})_k(\sigma_4)_k}{\Delta_1} &= Q_5 \frac{K_4 A_{44} Q_5 - \sqrt{K_4 K_5} A_{45} Q_4}{K_4 K_5 \Delta_2},
\end{aligned} \tag{20}$$

where

$$\Delta_1 = (\bar{Q}_{44})_k(\bar{Q}_{55})_k - (\bar{Q}_{45})_k(\bar{Q}_{45})_k, \qquad \Delta_2 = A_{44}A_{55} - A_{45}A_{45}. \tag{21}$$

Since the energy principle can be employed to derive the gross equilibrium equations, the use of shear strain energy equivalence formulations (20) and (21) to determine the values of K_m is a reasonable and consistent approach.

Modified 3-D equilibrium equations

The quasi-3-D shear stresses in the energy equivalence formulations are obtained from the 3-D equilibrium equations. In an orthogonal curvilinear coordinate system, the 3-D equilibrium equations are (Leipholz, 1974)

$$\sum_{j=1}^{3} (h_1 h_2 h_3 h_i \sigma_{ij}/h_j)_{,j} - \sum_{j=1}^{3} h_1 h_2 h_3 (h_j)_{,i} \sigma_{jj}/h_j = 0, \quad i = 1,2,3\,; \text{not summed.} \tag{22}$$

In the shell coordinate system shown in Fig. 1, the scale factors h_i in the above equations are

$$h_1 = A_1(1+z/R_1), \quad h_2 = A_2(1+z/R_2), \quad h_3 = 1 \tag{23}$$

and the first two equations ($i = 1, 2$) of eqns (22) can be written explicitly as follows:

$$\frac{R_1}{R_1+z} A_2 \sigma_{11,1} + \frac{R_2}{R_2+z} A_1 \sigma_{12,2} + A_1 A_2 \sigma_{13,3} + \frac{R_2}{R_2+z} A_{2,1}(\sigma_{11} - \sigma_{22}) + \frac{2R_1}{R_1+z} A_{1,2} \sigma_{12} + \left(\frac{2}{R_1+z} + \frac{1}{R_2+z}\right) A_1 A_2 \sigma_{13} = 0 \tag{24}$$

$$\frac{R_1}{R_1+z} A_2 \sigma_{12,1} + \frac{R_2}{R_2+z} A_1 \sigma_{22,2} + A_1 A_2 \sigma_{23,3} + \frac{R_1}{R_1+z} A_{1,2}(\sigma_{22} - \sigma_{11}) + \frac{2R_2}{R_2+z} A_{2,1} \sigma_{12} + \left(\frac{2}{R_2+z} + \frac{1}{R_1+z}\right) A_1 A_2 \sigma_{23} = 0. \tag{25}$$

With stresses (σ_{11}, σ_{12}, σ_{22}) calculated from the shell theory, it is not difficult to show that transverse shear stress distributions (σ_{23} and σ_{13}) along the thickness direction can be obtained through the integration of eqns (24) and (25).

For an N-layer laminate, the integration of eqns (24) or (25) over the thickness yields a total of N unknown coefficients. However, the number of boundary traction conditions and interlaminar stress continuity requirements is $(N+1)$. Therefore, shear stresses obtained from eqns (24) and (25) do not always satisfy both boundary conditions and continuity requirements. Here, the following modified equations

$$(A_2\sigma_{11})_{,1} + (A_1\sigma_{12})_{,2} + A_1 A_2 \sigma_{13,3} + A_{1,2}\sigma_{12} \underline{+ A_1 A_2 \sigma_{13}/R_1} - A_{2,1}\sigma_{22} = 0 \tag{26}$$

$$(A_2\sigma_{12})_{,1} + (A_1\sigma_{22})_{,2} + A_1 A_2 \sigma_{23,3} - A_{1,2}\sigma_{11} \underline{+ A_1 A_2 \sigma_{23}/R_2} + A_{2,1}\sigma_{12} = 0 \tag{27}$$

are employed to determine the consistent shear stress distributions for the shell analysis. In the cases of the FSDT ($c = 0$) and shallow shell theory (with underlined terms neglected), by imposing the zero shear traction condition on either surface, the zero traction condition on the other surface will be satisfied automatically. This is because the integrations of eqns (26) and (27) through the whole thickness will reproduce shell equations (11) and (12), and these two equations have been satisfied in the shell analysis. For the present theory, integrating the products of eqns (26) and (27) by factors

$$1 + cz^3/R_i, \quad i = 1 \text{ for eqn (26)}, \quad i = 2 \text{ for eqn (27)}$$

yields eqns (11) and (12), and the following integrals

$$\frac{cA_1A_2}{R_iR_i}\int \sigma_{i3}z^3\,\mathrm{d}z, \quad i = 1,2;\text{ not summed.}$$

Therefore, the quasi-3-D shear stresses calculated from eqns (26) and (27) nearly satisfy both the traction condition and the continuity requirement.

Iterative shear strain energy equivalence formulation

The values of K_m can be determined from eqns (20) and (21) if stresses $(\sigma_m)_k$ can be expressed in terms of stress resultants Q_m. Unfortunately, it seems that this condition does not exist in most structural problems. To overcome the difficulty, an iterative version of eqns (20) is employed to determine the shear correction factors for laminates having arbitrary lay-up. Let

$$\sum_{k=1}^{N}\int_{z_{k-1}}^{z_k}(\sigma_4)_k^{(i)}\frac{(\bar{Q}_{55})_k(\sigma_4)_k^{(i)}-(\bar{Q}_{45})_k(\sigma_5)_k^{(i)}}{\Delta_1} = Q_4^{(i)}\frac{K_5^{(i+1)}A_{55}Q_4^{(i)}-\sqrt{K_4^{(i+1)}K_5^{(i+1)}}A_{45}Q_5^{(i)}}{K_4^{(i+1)}K_5^{(i+1)}\Delta_2}$$

$$\sum_{k=1}^{N}\int_{z_{k-1}}^{z_k}(\sigma_5)_k^{(i)}\frac{(\bar{Q}_{44})_k(\sigma_5)_k^{(i)}-(\bar{Q}_{45})_k(\sigma_4)_k^{(i)}}{\Delta_1} = Q_5^{(i)}\frac{K_4^{(i+1)}A_{44}Q_5^{(i)}-\sqrt{K_4^{(i+1)}K_5^{(i+1)}}A_{45}Q_4^{(i)}}{K_4^{(i+1)}K_5^{(i+1)}\Delta_2}, \quad (28)$$

where (i) indicates the ith iteration. Initially the correction factors $K_m^{(0)}$ can be assigned arbitrary values. Upon solving the shell problem using eqns (11)–(15), the resultant forces $Q_m^{(0)}$ are determined and the quasi-3-D shear stresses $(\sigma_m)_k^{(0)}$ can be obtained from eqns (26) and (27). The improved values $K_m^{(1)}$ can then be calculated from eqns (28). The same procedure is repeated until convergence is achieved. It is shown in the examples presented hereafter that one iteration is usually sufficient to produce near-converged values.

The algorithm described by eqns (28) is particularly useful for numerical methods based upon a discretization scheme. However, the application of (28), used in conjunction with analytical procedure based on the continuum approach, is limited. This is because, in general, the values of shear correction factors vary over the laminate. That is, after the first iteration the laminate changes from a presumed homogeneous material (in the surface directions) into a heterogeneous material in the sense that stiffnesses S_{mn} are no longer uniform throughout the laminate. This poses a difficulty for subsequent analyses (using analytical methods). One exception to the above limitation occurs when cross-ply laminates admit series-type closed form solutions (Navier solutions). The shear correction factors corresponding to each individual mode of the series expansions are constant. Therefore, formulations (28) are seen to be suitable for such problems.

NUMERICAL EXAMPLES AND DISCUSSIONS

The shear correction factors and corresponding structural responses of symmetric and antisymmetric cross-ply laminated shells (including plates) are calculated. For these particular types of laminates, both shear stiffnesses $(\bar{Q}_{45})_k$ and A_{45} vanish. Therefore, iterative formulations (28) are reduced to

$$\sum_{k=1}^{N}\int_{z_{k-1}}^{z_k}\frac{(\sigma_m)_k^{(i)}(\sigma_m)_k^{(i)}}{(\bar{Q}_{mm})_k}\mathrm{d}z = \frac{Q_m^{(i)}Q_m^{(i)}}{K_m^{(i+1)}A_{mm}}, \quad m = 4,5;\text{ not summed.} \quad (29)$$

It is noted that K_4 and K_5 can be determined separately from the above formulations.

The laminates considered have rectangular platforms. The surface coordinates are denoted as x and y instead of ξ_1 and ξ_2. The Lamé parameters A_1 and A_2 are taken to be unity, i.e. $(\mathrm{d}s)^2 = (\mathrm{d}x)^2+(\mathrm{d}y)^2$. It is important to mention that this metric equation leads to approximate formulations in the cases of shallow, doubly curved surfaces. The edge widths of the middle surface along the x and y directions are a and b, respectively. The constant radii of curvatures of the middle surface in the x and y directions are denoted by R_x and R_y, respectively.

The laminates are constructed regularly in the sequence of [0°/90°/0°/...] with equal layer thickness. The following lamina material properties are used (Pagano, 1970)

$$E_1 = 25E_2, \quad E_3 = E_2, \quad G_{12} = G_{13} = 0.5E_2, \quad G_{23} = 0.2E_2, \qquad \nu_{12} = \nu_{13} = \nu_{23} = 0.25. \tag{30}$$

The upper surfaces of the laminates are subjected to the sinusoidally distributed loading, $p^+ = p \sin(\pi x/a) \sin(\pi y/b)$.

The laminates considered are simply supported in such a manner that the edges are fixed against tangential displacements but free to translate in the normal direction. In the shell analysis, the specified kinematic conditions along the edges are

$$u_s^0 = w^0 = \theta_s = 0 \tag{31}$$

and the force conditions are

$$N_n = M_n = P_n = 0, \tag{32}$$

where subscripts n and s indicate the normal direction to the edge and the tangential direction along the edge. The edge's kinematic conditions of the one of the corresponding 3-D problems are

$$u_s = w = 0. \tag{33}$$

Closed-form solutions of shell problems with boundary conditions (31) and (32) can be obtained. The solution algorithms are analogous to those of Reddy's plate and shallow shell problems (Reddy, 1984; Reddy and Liu, 1985). The solutions for displacements have the form

$$(u^0, \theta_x) = (\hat{u}^0, \hat{\theta}_x) \cos\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{b}\right)$$

$$w^0 = \hat{w}^0 \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{b}\right)$$

$$(v^0, \theta_y) = (\hat{v}^0, \hat{\theta}_y) \sin\left(\frac{\pi x}{a}\right) \cos\left(\frac{\pi y}{b}\right).$$

The details will not be provided here. Elasticity solutions to the 3-D shell problems and plate problems with boundary conditions (33) can also be obtained. Solution strategies listed in the literature (Pagano, 1970; Huang and Tauchert, 1991, 1992) are employed to obtain the exact solutions for the purpose of comparison.

Due to the simplification of the displacement field in the shell theory (especially deflection w), the relation between the 2-D problems and 3-D problems is seldom one-to-one. For example, the structural response of a solid shell with edges fixed against transverse deflection over the whole thickness [as described in eqn (33)] is identical to the predicted response of the same shell fixed against deflection along the center line of the edges with following kinematic boundary conditions

$$u_s = w^0 = 0 \tag{34}$$

when analyzed using the present shell theory. This is because these two different supporting conditions [eqns (33) and (34)] for the 3-D problems correspond to the same boundary conditions [eqns (31) and (32)] for the shell problems. This causes certain difficulty and controversy in evaluating the shell theories for a moderately thick laminate having low transverse elastic stiffnesses. In this respect, the elasticity solutions corresponding to the boundary conditions (33) can only provide approximate "lower-bound" deflection magnitudes for assessing the accuracy of the deflection responses predicted using the shell theories.

Closed-form solutions of the stresses and transverse shear forces in shell problems having boundary conditions (31) and (32) are given by

$$(\sigma_{xx}, \sigma_{yy}) = (\hat{\sigma}_{xx}(z), \quad \hat{\sigma}_{yy}(z)) \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{b}\right) \tag{35}$$

$$\sigma_{xy} = \hat{\sigma}_{xy}(z) \cos\left(\frac{\pi x}{a}\right) \cos\left(\frac{\pi y}{b}\right) \tag{36}$$

$$Q_{yz} = \hat{Q}_{yz} \sin\left(\frac{\pi x}{a}\right) \cos\left(\frac{\pi y}{b}\right) \tag{37}$$

$$Q_{xz} = \hat{Q}_{xz} \cos\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{b}\right). \tag{38}$$

The quasi-3-D stresses, obtained by substituting σ_{xx}, σ_{yy} and σ_{xy} in eqns (35) and (36) into eqns (26) and (27), take the form

$$\sigma_{yz} = \hat{\sigma}_{yz}(z) \sin\left(\frac{\pi x}{a}\right) \cos\left(\frac{\pi y}{b}\right) \tag{39}$$

$$\sigma_{xz} = \hat{\sigma}_{xz}(z) \cos\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{b}\right). \tag{40}$$

The shear correction factors K_m can then be determined from iterative formulations (29). It is noted that K_m are uniform over the laminate for this special loading condition.

Cylindrical laminates (including plates)

First, consider rectangular ($b/a = 3$) laminated cylindrical panels and plates. The convergence rate of shear correction factors, calculated from iterative formulations (29), of a five-layer cylindrical laminate with $R_x/a = 2$ and $a/h = 5$, are shown in Table 1. The corresponding center deflections w, calculated using shell theories (w_{sh}) and elasticity theory (w_{el}), are also listed in this table. The initial values of K_4 and K_5 are taken to be 1, which lead to the same results predicted by using the Reddy theory in the case where the shallow shell theory is employed. A rapid convergence rate is observed; a single iteration gives

Table 1. Convergence rate of shear correction factors and center deflections of a five-layer cylindrical laminate ($R_x/a = 2$, $b/a = 3$, $a/h = 5$)

Iteration No.	(K_4, K_5)†	$\bar{w}_{sh}$†	(K_4, K_5)‡	$\bar{w}_{sh}$‡	$\bar{w}_{el}$
0	(1, 1)	2.036	(1, 1)	1.936	
1	(0.7753, 0.7962)	2.375	(0.7759, 0.7959)	2.260	2.333
2	(0.7744, 0.7951)	2.377	(0.7750, 0.7948)	2.264	
3	(0.7744, 0.7951)	2.377	(0.7750, 0.7948)	2.264	

$(\bar{w}_{sh}, \bar{w}_{el}) = (w_{sh}(a/2, b/2, 0), w_{el}(a/2, b/2, 0))/h \times E_2/p \times (h/a)^4 \times 100$.
† Present theory.
‡ Shallow shell theory.

satisfactory results of K_m and deflections. It is also noted that the HSDT coupled with shear correction factors yields more accurate deflections than does the use of the HSDT without resort to the shear correction factors (letting $K_m = 1$).

The elastic responses, predicted by shell and elasticity theories, of laminates having various number (N) of layers, and various ratios of R_x/a and a/h, are reported in Tables 2–5. Table 2 shows the center deflections of the cross-ply laminates. The deflections on the upper surface of some moderately thick laminates predicted from the elasticity theory are also included in this table. For the laminates with $a/h = 10$, the variation of the deflection magnitude along the thickness is seen to be negligibly small. For the laminates with $R_x/a = 1$, deflections predicted using both the present theory and the shallow shell formulations are reported. A large discrepancy between the shallow shell solution and the elasticity solution is observed. However, for the shallow, thin shells ($R_x/a = 4$, $a/h = 50$), the deflection responses corresponding to the shallow shell theory are close to the elasticity solutions.

Table 2. Nondimensional center deflections, $\bar{w} = w(a/2, b/2, 0)/h \times E_2/p \times (h/a)^4 \times 100$, of laminated cylindrical panels and plates ($b/a = 3$)

R_x/a	a/h	Theory	Layer No. $N = 2$	$N = 3$	$N = 4$	$N = 5$	$N = 10$
1	5	Elast.	(4.381)	(2.727)	(3.720)	(2.826)	(3.311)
		Elast.	4.392	2.716	3.707	2.818	3.300
		Shell†	4.308	2.525	3.109	2.458	2.918
		Shell‡	4.463	2.758	3.804	2.859	3.330
		Shell§	3.753	2.053	2.606	2.001	2.410
		Shell¶	3.988	2.247	3.278	2.333	2.790
	10	Elast.	(2.964)	(1.149)	(1.845)	(1.237)	(1.622)
		Elast.	2.977	1.153	1.851	1.242	1.623
		Shell†	2.954	1.077	1.685	1.144	1.531
		Shell‡	2.994	1.149	1.861	1.238	1.627
		Shell§	2.563	0.8773	1.413	0.9339	1.267
		Shell¶	2.607	0.9370	1.571	1.012	1.352
	50	Elast.	0.7286	0.4082	0.5672	0.4436	0.5356
		Shell†	0.7287	0.4060	0.5659	0.4414	0.5343
		Shell‡	0.7289	0.4073	0.5673	0.4427	0.5352
		Shell§	0.6964	0.3536	0.5184	0.3890	0.4845
		Shell¶	0.6966	0.3548	0.5199	0.3902	0.4855
4	5	Elast.	(3.749)	(2.161)	(3.075)	(2.245)	(2.686)
		Elast.	3.745	2.118	3.042	2.205	2.648
		Shell§	3.641	1.944	2.494	1.896	2.298
		Shell¶	3.871	2.131	3.155	2.217	2.669
	10	Elast.	2.783	0.9396	1.609	1.020	1.381
		Shell§	2.750	0.8712	1.441	0.9308	1.283
		Shell¶	2.799	0.9323	1.612	1.011	1.373
	50	Elast.	2.139	0.5129	1.043	0.6059	0.9113
		Shell§	2.121	0.5041	1.027	0.5956	0.8981
		Shell¶	2.123	0.5066	1.033	0.5986	0.9013
Plate	5	Elast.	(3.713)	(2.102)	(3.015)	(2.183)	(2.626)
		Elast.	3.705	2.051	2.976	2.135	2.576
		Shell†	3.570	1.899	2.447	1.852	2.246
		Shell‡	3.794	2.082	3.088	2.166	2.610
	10	Elast.	2.776	0.9189‖	1.589	0.9981	1.357
		Shell†	2.743	0.8622	1.430	0.9214	1.272
		Shell‡	2.792	0.9235	1.601	1.001	1.361
	50	Elast.	2.475	0.5205‖	1.106	0.6195	0.9550
		Shell†	2.474	0.5179	1.099	0.6164	0.9515
		Shell‡	2.476	0.5205	1.106	0.6196	0.9551

Figures in parentheses indicate elasticity results sampled on $z = h/2$.
† HSDT with $K_m = 1$.
‡ HSDT with K_m calculated from eqns (29).
§ Shallow HSDT with $K_m = 1$.
¶ Shallow HSDT with K_m calculated from eqns (29).
‖ Also see Pagano (1970).

Table 3. Nondimensional displacements, $\bar{u} = u(0,\ b/2,\ z)/h \times E_2/p \times (h/a)^3 \times 100$ and $\bar{v} = v(a/2,\ 0,\ z)/h \times E_2/p \times (h/a)^3 \times 100$, of laminated cylindrical panels and plates ($b/a = 3$)

R_x/a	a/h	Theory	$\bar{u}$ $N = 3$	$\bar{u}$ $N = 4$	$\bar{u}$ $N = 5$	$\bar{v}$ $N = 3$	$\bar{v}$ $N = 4$	$\bar{v}$ $N = 5$
1	5	Elast.	5.831^-	7.835^-	6.000^-	-1.823^+	-1.620^+	-1.742^+
		Shell†	5.406^-	6.849^-	5.376^-	-1.678^+	-1.473^+	-1.532^+
		Shell‡	5.825^-	8.081^-	6.080^-	-1.798^+	-1.752^+	-1.739^+
	10	Elast.	4.707^-	7.374^-	5.097^-	-1.012^+	-1.175^+	-1.025^+
		Shell†	4.427^-	6.855^-	4.783^-	-0.9328^+	-1.091^+	-0.9398^+
		Shell‡	4.670^-	7.439^-	5.095^-	-0.9862^+	-1.199^+	-1.009^+
4	5	Elast.	2.100^-	2.915^-	2.155^-	-1.084^+	1.539^-	-1.069^+
		Shell§	1.930^-	2.641^-	1.976^-	-1.028^+	1.247^-	-0.9504^+
		Shell¶	2.049^-	3.052^-	2.167^-	-1.117^+	1.497^-	-1.083^+
	10	Elast.	1.681^-	2.701^-	1.855^-	-0.5568^+	0.8849^-	-0.5870^+
		Shell§	1.579^-	2.542^-	1.760^-	-0.5215^+	0.7911^-	-0.5418^+
		Shell¶	1.643^-	2.715^-	1.840^-	-0.5563^+	0.8662^-	-0.5838^+
Plate	5	Elast.	1.222^-	-3.663^+	-1.251^+	1.003^-	1.624^-	0.9942^-
		Shell†	1.130^-	-2.524^+	-1.195^+	0.9202^-	1.310^-	0.8554^-
		Shell‡	1.175^-	-2.728^+	-1.257^+	1.000^-	1.578^-	0.9727^-
	10	Elast.	0.9197^-	-2.536^+	1.029^-	0.4746^-	0.9879^-	0.5069^-
		Shell†	0.8779^-	-2.206^+	1.011^-	0.4401^-	0.8843^-	0.4638^-
		Shell‡	0.8926^-	-2.258^+	1.026^-	0.4697^-	0.9711^-	0.4993^-

† HSDT with $K_m = 1$.
‡ HSDT with K_m calculated from eqns (29).
§ Shallow HSDT with $K_m = 1$.
¶ Shallow HSDT with K_m calculated from eqns (29).
$^+$ Values calculated on $z = h/2$.
$^-$ Values calculated on $z = -h/2$.

Table 2 reveals that the improvement of the HSDT incorporated with shear correction factors in predicting the deflections is significant for the laminates having a large number of layers ($N = 4$, 5, 10). For example, for the case of a five-layer laminate with $R_x/a = 1$ and $a/h = 5$ the ratio of center deflections between the shell solution with $K_m = 1$ and the elasticity solution is equal to 87.22%; meanwhile the ratio between the shell solution with K_m calculated from eqns (29) and the elasticity solution is 101.5%. For the cases of two-layer laminates, the HSDT with $K_m = 1$ and K_m calculated from eqns (29) produce fairly accurate results; the discrepancy between shell solutions and elasticity solutions is very small for all the cases considered.

The in-plane displacements (u, v) predicted from shell theories and elasticity theory are shown in Table 3. Displacements are sampled on the upper surface ($z = h/2$) or lower surface ($z = -h/2$), at which their maximum values occur. It is observed that the HSDT theory coupled with the shear correction factors gives better results than the HSDT without shear correction factors. In-plane stresses (σ_{xx}, σ_{yy}, σ_{xy}) are reported in Table 4. The stresses are calculated at locations where their maximum values or near-maximum values occur. Comparing the shell solutions to the elasticity solutions, it is noted that the shell theory with $K_m = 1$ and K_m calculated from the energy equivalence formulation yields acceptable results. The improvement of the HSDT coupled with shear correction factors is more significant in predicting stresses σ_{yy} and σ_{xy} than in predicting σ_{xx}. The transverse quasi-3-D shear stresses, calculated from eqns (26) and (27), are reported in Table 5. It is seen that shell solutions are close to the elasticity solutions.

Spherical laminates (including plates)

The effect of shear correction factors on the deflection responses of spherical laminates ($R_x = R_y = R$) and plates is also investigated. The nondimensional center deflections of laminates having various numbers of layers and various ratios of R/a and a/h are shown in Table 6. For laminates with $a/h = 5$, the deflections on the upper surface are also included in this table to show the variation of the deflections in the thickness direction. The effect of

Table 4. Nondimensional stresses $(\bar{\sigma}_{xx}, \bar{\sigma}_{yy}, \bar{\sigma}_{xy}) = (\sigma_{xx}(a/2, b/2, z), \sigma_{yy}(a/2, b/2, z), \sigma_{xy}(0, 0, z))/p \times (h/a)^2$ of laminated cylindrical panels and plates $(b/a = 3)$

R_x/a	a/h	Theory	$\bar{\sigma}_{xx}$ $N = 3$	$\bar{\sigma}_{xx}$ $N = 4$	$\bar{\sigma}_{xx}$ $N = 5$	$\bar{\sigma}_{yy} \times 10$ $N = 3$	$\bar{\sigma}_{yy} \times 10$ $N = 4$	$\bar{\sigma}_{yy} \times 10$ $N = 5$	$\bar{\sigma}_{xy} \times 10$ $N = 3$	$\bar{\sigma}_{xy} \times 10$ $N = 4$	$\bar{\sigma}_{xy} \times 10$ $N = 5$
1	5	Elast.	-1.293^-	-1.668^-	-1.301^-	2.411‖	4.672^+	3.132‖	0.4371^-	0.6597^-	0.4561^-
		Shell†	-1.094^-	-1.500^-	-1.154^-	2.228‖	4.091^+	2.784‖	0.4046^-	0.5475^-	0.3983^-
		Shell‡	-1.132^-	-1.600^-	-1.207^-	2.387‖	4.838^+	3.139‖	0.4325^-	0.6389^-	0.4479^-
	10	Elast.	-0.8534^-	-1.222^-	-0.9436^-	1.602‖	3.314^+	2.044‖	0.2725^-	0.4883^-	0.3030^-
		Shell†	-0.7876^-	-1.176^-	-0.9000^-	1.498‖	3.049^+	1.895‖	0.2581^-	0.4478^-	0.2839^-
		Shell‡	-0.7982^-	-1.194^-	-0.9098^-	1.586‖	3.333^+	2.030‖	0.2712^-	0.4842^-	0.3019^-
4	5	Elast.	-1.022^-	-1.388^-	-1.040^-	1.116‖	3.117^+	1.763‖	0.2588^-	0.4006^-	0.2626^-
		Shell§	-0.9125^-	-1.301^-	-0.9641^-	1.028‖	2.640^+	1.547‖	0.2356^-	0.3342^-	0.2293^-
		Shell¶	-0.9484^-	-1.418^-	-1.014^-	1.112‖	3.209^+	1.741‖	0.2533^-	0.3952^-	0.2562^-
	10	Elast.	-0.7463^-	-1.137^-	-0.8340^-	0.6468‖	2.045^+	1.047‖	0.1510^-	0.2822^-	0.1660^-
		Shell§	-0.6985^-	-1.100^-	-0.8037^-	0.6037‖	1.846^+	0.9623‖	0.1405^-	0.2573^-	0.1543^-
		Shell¶	-0.7101^-	-1.129^-	-0.8160^-	0.6421‖	2.038^+	1.031‖	0.1476^-	0.2782^-	0.1630^-
Plate	5	Elast.	0.9835^+	-1.325^-	1.000^+	−0.9023††	2.809^+	−1.528††	0.2215^-	0.3429^-	0.2214^-
		Shell†	0.8924^+	-1.275^-	0.9430^+	−0.7843††	2.378^+	−1.320††	0.2037^-	0.2903^-	0.1969^-
		Shell‡	0.9277^+	-1.391^-	0.9926^+	−0.8468††	2.881^+	−1.477††	0.2187^-	0.3401^-	0.2186^-
	10	Elast.	0.7260^+‡‡	-1.116^-	0.8120^+	−0.4349††‡‡	1.740^+	−0.8297††	0.1227^-‡‡	0.2292^-	0.1335^-
		Shell†	0.6924^+	-1.094^-	0.7969^+	−0.3981††	1.581^+	−0.7621††	0.1151^-	0.2115^-	0.1258^-
		Shell‡	0.7040^+	-1.124^-	0.8094^+	−0.4227††	1.743^+	−0.8139††	0.1205^-	0.2271^-	0.1322^-

† HSDT with $K_m = 1$.
‡ HSDT with K_m calculated from eqns (29).
§ Shallow HSDT with $K_m = 1$.
¶ Shallow HSDT with K_m calculated from eqns (29).
‖ Values calculated on the upper face of the second layer.
†† Values calculated on the lower face of the second layer.
‡‡ Also see Pagano (1970).
$^+$ Values calculated on $z = h/2$.
$^-$ Values calculated on $z = -h/2$.

Table 5. Nondimensional transverse shear stresses $\bar{\sigma}_{xz} = \sigma_{xz}(0, b/2, 0)/p \times (h/a)$ and $\bar{\sigma}_{yz} = \sigma_{yz}(a/2, 0, 0)/p \times (h/a)$ of laminated cylindrical panels and plates ($b/a = 3$)

R_x/a	a/h	Theory	$\bar{\sigma}_{xz}$ $N = 3$	$N = 4$	$N = 5$	$\bar{\sigma}_{yz} \times 10$ $N = 3$	$N = 4$	$N = 5$
1	5	Elast.	0.4447	0.5392	0.4831	0.3442	0.8547	0.5104
		Shell†	0.4794	0.5946	0.4564	0.3169	0.7233	0.4498
		Shell‡	0.4729	0.5793	0.4471	0.3304	0.8117	0.4906
	10	Elast.	0.4697	0.5597	0.4544	0.1848	0.5567	0.2904
		Shell†	0.4821	0.5781	0.4491	0.1746	0.5028	0.2679
		Shell‡	0.4790	0.5701	0.4456	0.1798	0.5334	0.2803
4	5	Elast.	0.3867	0.4924	0.4260	0.2729	0.7049	0.4016
		Shell§	0.4118	0.5443	0.3922	0.2479	0.5995	0.3511
		Shell¶	0.4068	0.5336	0.3858	0.2638	0.6939	0.3857
	10	Elast.	0.4271	0.5379	0.4160	0.1555	0.4869	0.2425
		Shell§	0.4344	0.5525	0.4068	0.1462	0.4400	0.2235
		Shell¶	0.4068	0.5492	0.4050	0.1524	0.4756	0.2349
Plate	5	Elast.	0.3755	0.4837	0.4094	0.1324	0.3450	0.3892
		Shell†	0.4022	0.5321	0.3833	0.1210	0.2949	0.3429
		Shell‡	0.3973	0.5220	0.3769	0.1290	0.3416	0.3769
	10	Elast.	0.4201‖	0.5333	0.4093	0.1524‖	0.4801	0.2375
		Shell†	0.4299	0.5480	0.4027	0.1447	0.4407	0.2213
		Shell‡	0.4283	0.5451	0.4010	0.1508	0.4734	0.2326

† HSDT with $K_m = 1$.
‡ HSDT with K_m calculated from eqns (29).
§ Shallow HSDT with $K_m = 1$.
¶ Shallow HSDT with K_m calculated from eqns (29).
‖ Also see Pagano (1970).

the shear correction factors on the shell solutions is more significant for the laminates having a large number of layers. For example, considering laminates with $R/a = 5$ and $a/h = 10$ having various layer numbers $N = 2, 3, 4, 5, 10$, the ratios of center deflections between shallow shell solutions without resort to the shear correction factors ($K_m = 1$) and elasticity solutions are 98.4%, 94.3%, 90.0%, 92.2% and 93.1%, respectively. However, the ratios between shallow shell solutions with shear correction factors and elasticity solutions are 99.7%, 99.9%, 100.1%, 99.8% and 99.8%, respectively.

CONCLUDING REMARKS

A laminated shell theory based on the orthogonal curvilinear coordinates has been presented. Upon applying the Donnell shallow shell approximation, the theory leads to the Reddy formulation in the case where the laminate has a Euclidean middle surface and constant principal curvatures. Shear correction factors are introduced in the present higher order theory to accommodate the effect of the continuity of the interlaminar shear stresses. The shear correction factors are calculated from an iterative formulation based on the shear strain energy equivalence.

The effect of shear correction factors upon the structural responses predicted using the HSDT was examined by comparing the shell solutions to the elasticity solutions. The influence of shear correction factors is significant on the deflection behaviour of moderately thick laminates having a large number of layers. For the cases of two- and three-layer laminates, the HSDT predicts fairly accurate structural responses without resort to the shear correction factors.

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Table 6. Nondimensional center deflections, $\bar{w} = w(a/2, b/2, 0)/h \times E_2/p \times (h/a)^4 \times 100$, of laminated spherical panels and plates ($R_x = R_y = R$, $b/a = 1$)

R/a	a/h	Theory	Layer No. $N = 2$	$N = 3$	$N = 4$	$N = 5$	$N = 10$
2	5	Elast.	(1.664)	(1.510)	(1.465)	(1.399)	(1.317)
		Elast.	1.646	1.482	1.434	1.376	1.284
		Shell†	1.615	1.420	1.228	1.217	1.148
		Shell‡	1.683	1.519	1.498	1.404	1.315
		Shell§	1.551	1.363	1.177	1.167	1.101
		Shell¶	1.618	1.459	1.439	1.348	1.262
	10	Elast.	0.8533	0.6087	0.6128	0.5671	0.5495
		Shell†	0.8480	0.5840	0.5673	0.5344	0.5221
		Shell‡	0.8567	0.6119	0.6170	0.5696	0.5516
		Shell§	0.8207	0.5633	0.5471	0.5151	0.5031
		Shell¶	0.8292	0.5905	0.5956	0.5494	0.5319
5	5	Elast.	(1.761)	(1.585)	(1.533)	(1.456)	(1.362)
		Elast.	1.736	1.549	1.495	1.417	1.320
		Shell§	1.684	1.461	1.240	1.228	1.151
		Shell¶	1.762	1.573	1.553	1.443	1.340
	10	Elast.	1.157	0.7325	0.7408	0.6707	0.6451
		Shell§	1.139‖	0.6905‖	0.6664	0.6182	0.6008
		Shell¶	1.153	0.7318	0.7418	0.6696	0.6436
Plate	5	Elast.	(1.744)	(1.567)	(1.516)	(1.439)	(1.345)
		Elast.	1.712	1.525	1.471	1.392	1.296
		Shell†	1.667††	1.442	1.218	1.207	1.129††
		Shell‡	1.745	1.554	1.535	1.423	1.319
	10	Elast.	1.227	0.7530	0.7624	0.6866	0.6592
		Shell†	1.216‖††	0.7125‖	0.6966	0.6346	0.6160††
		Shell‡	1.232	0.7572	0.7685	0.6900	0.6619

Figures in parentheses indicate elasticity results sampled on $z = h/2$.
† HSDT with $K_m = 1$.
‡ HSDT with K_m calculated from eqns (29).
§ Shallow HSDT with $K_m = 1$.
¶ Shallow HSDT with K_m calculated from eqns (29).
‖ Also see Reddy and Liu (1985) [multiplied by a factor $(1+h/R/2)^2$].
†† Also see Khdeir and Reddy (1991).

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